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THE HAMILTON-JACOBI FORMULATION OF THE RESTRICTED THREE BODY PROBLEM IN TERMS OF THE TWO FIXED CENTER PROBLEM

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FOREWORD

This document is the First Semiannual Report prepared by Republic Aviation Corporation under NASA Contract No. NAS8-2605, "Research Regarding Guidance and Space Flight Theory Relative to the Rendezvous Problem." The contract was initiated and is monitored by W. Miner and R. Hoelker of the Aeroballistics Laboratory, George C. Marshall Space Flight Center.

The document will appear in slightly different format as a part of PROGRESS REPORT NO. 2 ON STUDIES IN THE FIELDS OF SPACE FLIGHT AND GUIDANCE THEORY, sponsored by the Aeroballistics Division of the Marshall Space Flight Center.

The report was prepared by Dr. Mary Payne and Mr. Samuel Pines of Republic's Applied Mathematics Section, Applied Research and Development Center. The authors wish to express their appreciation for many helpful discussions with Mr. Elie Lowy and Dr. George Nomicos and they especially want to thank Dr. John Morrison whose comments, from the inception of the problem, have been most illuminating.

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NOTATION

$\underline{R}$	Position vector of the vehicle relative to the barycenter in a coordinate system fixed in space
$\underline{R}_1$	Position vector of the vehicle relative to the earth
$\underline{R}_2$	Position vector of the vehicle relative to the moon
R'	Position vector of the vehicle relative to the barycenter in a rotating system
$\bar{\underline{R}}$	Position vector of the vehicle relative to the midpoint of the earth-moon line
r_1	Magnitude of $\underline{R}_1$
r_2	Magnitude of $\underline{R}_2$
Ω	Angular velocity vector of earth-moon system
ω	Magnitude of ω
μ	Gravitational constant times mass of the earth
μ'	Gravitational constant times mass of the moon
$\mathcal{L}$	Lagrangian function
$\underline{P}$	Momentum vector
H	Hamiltonian function
q_i	Generalized coordinates conjugate to p_i
Q_i	Generalized coordinates conjugate to P_i
p_i	Generalized momenta conjugate to q_i
P_i	Generalized momenta conjugate to Q_i

ψ_1	Time-dependent generating function
t	Time
H_1	Integrable part of the Hamiltonian
H_2	Disturbing function
h	Energy constant for H_1
W	Time-independent generating function
J_i	Action variables
w_i	Angle variables
ν_i	Frequencies for two fixed center problem
q_1, q_2	Elliptic coordinates (prolate spheroidal coordinates)
φ	Angle measured around x-axis (prolate spheroidal coordinates)
c	Half the distance between earth and moon
P_1, P_2, P_φ	Momenta conjugate to prolate spheroidal coordinates
x, y, z	Rectangular coordinates in a system with earth at (c, o, o), moon at (-c, o, o) and $\underline{\Omega}$ in the z direction
α	Angular momentum about the line of centers in the two fixed center problem
β	Third dynamical constant of motion of the two fixed center problem
$R^2(q_1)$	Fundamental quartic associated with q_1
$S^2(q_2)$	Fundamental quartic associated with q_2
u	Parameter in terms of which coordinates and time of the two fixed center problem are given

r_i	Roots of $R^2(q_1) = 0$
s_i	Roots of $S^2(q_2) = 0$
n_i	Coefficient of linear term in q_i contribution to time as a function of u
m_i	Coefficient of linear term in q_i contribution to φ as a function of u
$F_i(u)$	Periodic term in time due to q_i
$G_i(u)$	Periodic term in φ due to q_i
K_1	Quarter period of q_1 elliptic functions
K_2	Quarter period of q_2 elliptic functions
Q_h	Coordinate conjugate to h
Q_α	Coordinate conjugate to α
Q_β	Coordinate conjugate to β

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Farmingdale, L.I., New York

THE HAMILTON-JACOBI FORMULATION OF THE RESTRICTED THREE BODY PROBLEM IN TERMS OF THE TWO FIXED CENTER PROBLEM

By

Mary Payne

Samuel Pines

Summary

This report contains a development of the classical Hamilton-Jacobi perturbation techniques, applying the known solution of the Two Fixed Center Problem to the Restricted Three Body Problem.

SECTION I - INTRODUCTION

This report contains an outline of the development of a perturbation procedure for solving the restricted three body problem, using the solution of the two fixed center problem as an intermediate orbit. In the restricted problem, it is assumed that the two primary bodies move in circles about their center of mass, the barycenter. The primary bodies will be fixed in a coordinate system rotating with their angular velocity, so that the use of the two fixed center problem is immediately suggested. The two fixed center problem was first treated by Euler, who discovered that its equations of motion are separable in prolate spheriodial coordinates. A very complete discussion of the two fixed center problem has been given by Charlier(1). This treatment covers some of the same ground as this report. It is from the

Hamiltonian point of view and includes a discussion of the action and angle variables, and the way in which the two fixed center problem would be used as a basis for a perturbation theory for the restricted problem. The only thing missing from Charlier's treatment is an explicit solution of the two fixed center problem, which would be necessary for the actual application to the restricted problem. Formal expressions for the action and angle variables are obtained from a more modern point of view by Buchheim(2). Brief discussions of the two fixed center problem are given in many standard text books such as Whittaker(3), Landau and Lifschitz(4) and Wintner(5). The explicit solution of the two fixed center problem has been obtained by Pines and Payne(6). In the present report, this solution will be combined with a Hamiltonian development of the problem to show how perturbation equations for the restricted problem may be obtained. A different development has been carried out recently by Davidson and Schulz-Arenstorff(7). In this theory, the initial conditions of a two fixed center problem are used as parameters and a first order correction for the restricted problem is obtained in closed form. Second-order corrections are obtained by a numerical curve-fitting scheme.

In this report, Section II will contain a discussion of the restricted problem, and the way in which the two fixed center problem will be used. In Section III, the solution of the two fixed center problem will be outlined in sufficient detail for the determination of the action and angle variables, which is carried out in Sections IV and V. Finally in Section VI a summary will be given of the essential steps still necessary to obtain the solution of the restricted problem.

SECTION II - THE RESTRICTED PROBLEM

The equations of motion of the restricted problem are

$$\ddot{\underline{R}} = -\mu \frac{\underline{R}_1}{r_1^3} - \mu' \frac{\underline{R}_2}{r_2^3} \tag{1}$$

where $\underline{R}$ is the position vector of the vehicle in a coordinate system fixed in space, $\underline{R}_1$ and $\underline{R}_2$ are respectively the position vectors of the vehicle from earth and moon (with magnitudes r_1 and r_2), and μ and μ' are the gravitational constant times mass of the earth and moon, respectively. Since the barycenter (center of mass of earth and moon) may be regarded as a point fixed in space, the vector $\underline{R}$ will henceforth be regarded as relative to a system fixed in space with origin at the barycenter. The earth and moon are taken as moving in circles about the barycenter with angular velocity vector $\underline{\Omega}$. To use the two fixed center problem as an approximation to the restricted problem, it is necessary to write the equations of motion in a coordinate system in which the earth and moon are fixed. Such a system is one rotating about the barycenter with angular velocity $\underline{\Omega}$ relative to the fixed system. Denoting the position vector in the rotating system by $\underline{R}'$, the equations of motion (1) become

$$\ddot{\underline{R}}' = -\mu \frac{\underline{R}_1}{r_1^3} - \mu' \frac{\underline{R}_2}{r_2^3} - 2\,\underline{\Omega} \times \dot{\underline{R}}' - \underline{\Omega} \times (\underline{\Omega} \times \underline{R}') \tag{2}$$

It is readily shown that the Lagrangian for the equations of motion (2) is

$$\mathcal{L} = \frac{1}{2}\,\dot{R}'^2 + \underline{\Omega} \cdot \underline{R}' \times \dot{\underline{R}}' + \frac{1}{2}\,(\underline{\Omega} \times \underline{R}')^2 + \frac{\mu}{r_1} + \frac{\mu'}{r_2} \tag{3}$$

and hence the momentum vector conjugate to the position vector $\underline{R}'$ is given by

$$\underline{P} = \text{grad}_{\dot{\underline{R}}'} \mathcal{L} = \dot{\underline{R}}' + \underline{\Omega} \times \underline{R} \tag{4}$$

and the Hamiltonian for the problem is

$$H = \underline{P} \cdot \dot{\underline{R}}' - \mathcal{L} = \frac{1}{2} P^2 - \underline{\Omega} \cdot \underline{R}' \times \underline{P} - \frac{\mu}{r_1} - \frac{\mu'}{r_2} \tag{5}$$

and the Hamiltonian equations are

$$\dot{\underline{P}} = -\text{grad}_{R'} H = -\underline{\Omega} \times \underline{P} - \mu \frac{\underline{R}_1}{r_1^3} - \mu' \frac{\underline{R}_2}{r_2^3} \tag{6}$$

and

$$\dot{\underline{R}}' = \text{grad}_P H = \underline{P} - \underline{\Omega} \times \underline{R}' \tag{7}$$

It will be noted that Eq. (7) is equivalent to Eq. (4), and that if $\underline{P}$ is replaced using Eq. (4), then Eq. (6) will yield the equations of motion (2).

The solution of the restricted problem will be carried out by making use of a transformation theorem (Reference 1, Chapter 11 and Reference 12, pp 237 to 246) which states that if the Hamiltonian of a system is $H(q_i, p_i, t)$ with q_i and p_i canonically conjugate coordinates so that the Hamilton equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{8}$$

are satisfied and if $\psi(q_i, P_i, t)$ is any function, then the variables Q_i and P_i defined by

$$Q_i = \frac{\partial \psi}{\partial P_i} = Q_i(q_i, P_i, t), \quad p_i = \frac{\partial \psi}{\partial q_i} = p_i(q_i, P_i, t) \tag{9}$$

are canonical variables for a new Hamiltonian

$$\bar{H} = H + \frac{\partial \psi}{\partial t} \tag{10}$$

regarded as a function of Q_i, P_i and t, so that

$$\dot{Q}_i = \frac{\partial \overline{H}}{\partial P_i} \quad , \quad \dot{P}_i = - \frac{\partial \overline{H}}{\partial Q_i} \tag{11}$$

Now let the Hamiltonian be separated into two terms

$$H = H_1 (q_i, p_i) + H_2 (q_i, p_i, t) \tag{12}$$

with H_1 independent of the time and such that the partial differential equation

$$H_1 (q_i, \frac{\partial \psi_1}{\partial q_i}) + \frac{\partial \psi_1}{\partial t} = 0 \tag{13}$$

possesses a solution for ψ_1. It is seen that if the function ψ_1 is used in the transformation theorem then the Hamilton equations become

$$\begin{aligned} \dot{Q}_i &= \frac{\partial (H_1 + H_2 + \frac{\partial \psi_1}{dt})}{\partial P_i} = \frac{\partial H_2}{\partial P_i} \\ \dot{P}_i &= \frac{\partial (H_1 + H_2 + \frac{\partial \psi_1}{\partial t})}{\partial Q_i} = - \frac{\partial H_2}{\partial Q_i} \end{aligned} \tag{14}$$

by virtue of the defining Eq. (13) for ψ_1. Further, from Eq. (13), it is evident that, since H_1 is independent of time,

$$\psi_1 = - ht + W(q_i , P_i) \tag{15}$$

with

$$H_1 (q_i, \frac{\partial W}{\partial q_i}) - h = 0 \tag{16}$$

and the momenta P_i must be identified with the constants of integration of Eq. (13) and h, the separation constant for the time. This is not to be interpreted as meaning that the P_i are constants of the motion for the Hamiltonian H. If this were so, the right-hand sides of Eq. (14) would have to vanish. What the solution of Eq. (16) for W does is to specify a function of q_i and three new variables P_i.

This function may be used to invert Eqs. (9) to obtain q_i and p_i in terms of the new variables P_i and three others Q_i. These expressions for q_i and p_i may now be inserted in H_2 for use in Eqs. (14) from which Q_i and P_i may now be obtained as functions of time. The solution of the problem associated with H will then be given by substituting the solutions Q_i and P_i of Eqs. (14) in the expression for q_i and p_i.

To actually carry out the inversion of Eqs. (9) it must be noted that the functional form of ψ_1 does not depend on the disturbing function ultimately to be used. It depends rather on how the identification of the P_i is made with the integration constants arising in Eq. (16). The conventional procedure is to regard H_1 as the Hamiltonian of a new problem and identify the P_i with the action variables J_i of this new problem. The action variables are always three independent functions of the integration constants and hence are themselves constant for the problem associated with H_1. Once the functional relation between the P_i and the integration constants is determined, by identifying the P_i with the action variables J_i of H_1, the conjugate coordinates Q_i are defined by Eq. (9). It will always happen that P_i and Q_i so defined are constant if the Hamiltonian is H_1 because from Eq. (13)

$$\begin{aligned} \dot{Q}_i &= \frac{\partial \left(H_1 + \frac{\partial \psi_1}{\partial t}\right)}{\partial P_i} = 0 \\ J_i = \dot{P}_i &= -\frac{\partial \left(H_1 + \frac{\partial \psi_1}{\partial t}\right)}{\partial Q_i} = 0 \end{aligned} \tag{17}$$

Once the functional relation between q_i and p_i and Q_i and J_i is established, however, the problem associated with H_1 is no longer of interest. The disturbing function H_2 is expressed in terms of Q_i and J_i and the solution of the problem associated with H is obtained by integrating Eqs. (14).

A slightly different formulation of the problem is obtained if the time independent function W of Eq. (15) is used as the generating function of the transformation rather than ψ_1. The variables w_i conjugate to the action variables

J_i are the angle variables of the problem associated with H_1. The relations between the w_i and the Q_i are given by

$$w_i = \frac{\partial W}{\partial J_i} = \frac{\partial(\psi_1 + ht)}{\partial J_i} = Q_i + t\frac{\partial h}{\partial J_i} = \nu_i t + Q_i \tag{18}$$

with

$$\nu_i = + \frac{\partial h}{\partial J_i} \tag{19}$$

being functions of the action variables. The perturbation equations for these variables will be given, according to the transformation theorem, by

$$\begin{aligned} \dot{w}_i &= \frac{\partial\left(H_1 + H_2 + \frac{\partial W}{\partial t}\right)}{\partial J_i} = \frac{\partial H_2}{\partial J_i} + \nu_1 \\ \dot{J}_i &= - \frac{\partial\left(H_1 + H_2 + \frac{\partial W}{\partial t}\right)}{\partial w_i} = - \frac{\partial H_2}{\partial w_i} \end{aligned} \tag{20}$$

since W is independent of time and $H_1 = h$ depends only on the action variables and not on the angle variables. The advantage of using the angle variables rather than the Q_i is that it will always be possible to expand H_2 in a multiple Fourier series in the angle variables and eliminate its explicit dependence on time.

To use the two fixed center problem to solve the restricted problem, the Hamiltonian (5) for the restricted problem may be separated into terms H_1 and H_2 as follows:

$$H_1 = \frac{1}{2} p^2 - \frac{\mu}{r_1} - \frac{\mu'}{r_2} \tag{21}$$

$$H_2 = - \underline{\Omega} \cdot \underline{R}' \times \underline{P} \tag{22}$$

If H_1 is regarded as a Hamiltonian, the associated Hamilton equations are

$$\dot{\underline{R}}' = - \text{grad}_P H_1 = \underline{P} \tag{23}$$

and

$$\underline{\dot{P}} = - \operatorname{grad}_{R'} H_1 = - \mu \frac{\underline{R}_1}{r_1^{\,3}} - \mu' \frac{\underline{R}_2}{r_2^{\,3}} = \underline{\ddot{R}}' \tag{24}$$

These last equations are just the equations of motion for the two fixed center problem, so that H_1 is the Hamiltonian of the two fixed center problem. Thus the procedure will be first to find the action and angle variables of the two fixed center problem and then express the disturbing function

$$H_2 = - \underline{\Omega} \cdot \underline{R}' \times \underline{P} \tag{25}$$

in terms of these variables.

Before proceeding with the details of this procedure, it is desirable to make two further transformation of the coordinates. The first will be to a coordinate system with the origin at the midpoint of the earth-moon line with the earth and moon on the x-axis at (c,0,0) and (-c,0,0) respectively. The distance between the earth and moon is thus 2c. The z-axis will be taken in the direction of $\underline{\Omega}$. The only term in the Hamiltonian affected by this transformation is the $\underline{\Omega} \cdot \underline{R}' \times \underline{P}$ term in which $\underline{R}'$ is measured from the barycenter. From Figure I it is evident

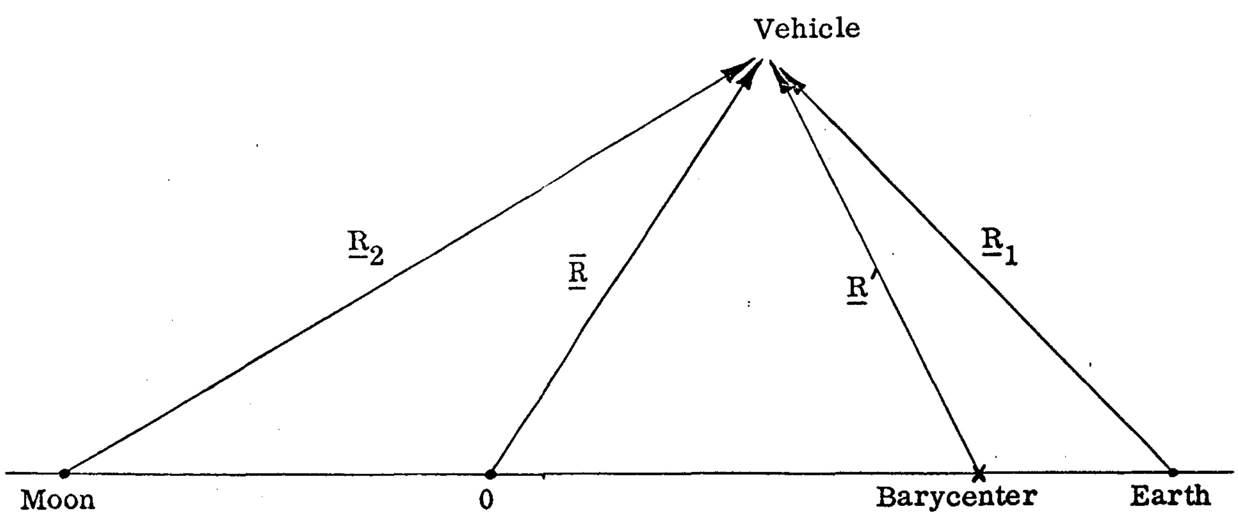


Figure I

that, since the barycenter is at the point $(c\,\frac{\mu - \mu'}{\mu + \mu'}, 0, 0)$ the position vectors of the vehicle relative to the midpoint are related by

$$\underline{R}' = \underline{\bar{R}} - ic\,\frac{\mu - \mu'}{\mu + \mu'} \tag{26}$$

where i is the unit vector in the x-direction. Thus the disturbing function becomes

$$H_2 = - \underline{\Omega} \cdot \underline{R}' \cdot \underline{P} = - \underline{\Omega} \cdot \underline{\bar{R}} \times \underline{P} + c\omega\,\frac{\mu - \mu'}{\mu + \mu'}\,(j \cdot \underline{P}) \tag{27}$$

where j is a unit vector in the y direction.

The second transformation will be from rectangular to prolate ellipsoidal coordinates, in which the Hamilton-Jacobi equation for the two fixed center problem is separable. This transformation may be effected by the generating function

$$F = c\,q_1 q_2\,P_x + c\sqrt{(q_1^2 - 1)(1 - q_2^2)}\,\cos\varphi\,P_y + c\sqrt{(q_1^2 - 1)(1 - q_2^2)}\,\sin\varphi\,P_z \tag{28}$$

with the new coordinates q_1, q_2, φ, p_1, p_2, and p_φ related to the old ones x, y, z, P_x, P_y, and P_z by

$$\begin{aligned} x &= \frac{\partial F}{\partial P_x} & p_1 &= \frac{\partial F}{\partial q_1} \\ y &= \frac{\partial F}{\partial P_y} & p_2 &= \frac{\partial F}{\partial q_2} \\ z &= \frac{\partial F}{\partial P_z} & p_\varphi &= \frac{\partial F}{\partial \varphi} \end{aligned} \tag{29}$$

From the equations for x, y, and z it is seen that

$$
\begin{aligned}
x &= c q_1 q_2 \\
y &= c \sqrt{(q_1^2 - 1)(1 - q_2^2)} \cos \varphi \\
z &= c \sqrt{(q_1^2 - 1)(1 - q_2^2)} \sin \varphi
\end{aligned}
\tag{30}
$$

In this system the surfaces $q_1 = \text{const} \geq 1$ are ellipsoids of revolution about the x-axis confocal about the earth and moon. The limiting surface $q_1 = 1$ is the portion of the x-axis between the earth and moon, and the ellipsoids increase in size with increasing q_1. The surfaces $-1 \leq q_2 = \text{const} \leq 1$ are hyperboloids of revolution about the x-axis, confocal about the earth and moon. The limiting surfaces $q_1 = 1$ and $q_2 = -1$ are the portions of the x-axis to the right of the earth and to the left of the moon, respectively. The surface $q_2 = 0$ is the y-z plane and surfaces corresponding to positive values of q_2 are hyperboloids concave towards the earth while those corresponding to negative values of q_2 are concave towards the moon. The angle φ is measured in the y-z plane about the x-axis and is zero in the portion of the x-y plane for which $y > 0$. From Eq. (30), it is easy to show that r_1 and r_2 which appear in the Hamiltonian (5) are given by

$$
\begin{aligned}
r_1 &= c (q_1 - q_2) \\
r_2 &= c (q_1 + q_2)
\end{aligned}
\tag{31}
$$

The equations for p_1, p_2, p_φ are

$$
\begin{aligned}
p_1 &= c q_2 P_x + \frac{c q_1 (1 - q_2^2) \cos\varphi}{\sqrt{(q_1^2 - 1)(1 - q_2^2)}} P_y + \frac{c q_1 (1 - q_2^2) \sin\varphi}{\sqrt{(q_1^2 - 1)(1 - q_2^2)}} P_z \\
p_2 &= c q_1 P_x - \frac{c q_2 (q_1^2 - 1) \cos\varphi}{\sqrt{(q_1^2 - 1)(1 - q_2^2)}} P_y - \frac{c q_2 (q_1^2 - 1) \sin\varphi}{\sqrt{(q_1^2 - 1)(1 - q_2^2)}} P_z
\end{aligned}
\tag{32}
$$

$$p_\varphi = -c\sqrt{(q_1^2-1)(1-q_2^2)}\,\sin\varphi\, P_y + c\sqrt{(q_1^2-1)(1-q_2^2)}\,\cos\varphi\, P_z \tag{32}$$

Inverting these equations to obtain P_x, P_y and P_z in terms of p_1, p_2 and p_φ one obtains for H_1

$$\begin{aligned} H_1 &= \frac{1}{2} P^2 - \frac{\mu}{r_1} - \frac{\mu'}{r_2} \\ &= \frac{1}{2c^2}\left[\frac{(q_1^2-1)\, p_1^2}{q_1^2 - q_2^2} + \frac{(1-q_2^2)\, p_2^2}{q_1^2 - q_2^2} + \frac{p_\varphi^2}{(q_1^2-1)(1-q_2^2)}\right] \\ &\qquad - \frac{u}{c\,(q_1 - q_2)} - \frac{\mu'}{c\,(q_1+q_2)} \end{aligned} \tag{33}$$

and for the disturbing function

$$\begin{aligned} H_2 = \omega \Bigg\{ &\frac{\sqrt{(q_1^2-1)(1-q_2^2)}}{q_1^2 - q_2^2} \cos\varphi \left[p_1 q_2 - p_2 q_1 + \frac{\mu-\mu'}{\mu+\mu'} (p_1 q_1 - p_2 q_2)\right] \\ &- \frac{p_\varphi \sin\varphi}{\sqrt{(q_1^2-1)(1-q_2^2)}} \left(q_1 q_2 + \frac{\mu - \mu'}{\mu + \mu'}\right) \Bigg\} \end{aligned} \tag{34}$$

This completes the preliminary discussion of the problem. The following sections contain the solution of the two fixed center problem which will be useful in the subsequent determination of the generating function W from Eq. (16) and the action and angle variables for the two fixed center problem which will be the w_i and J_i of the perturbation Eqs. (19).

SECTION III - SOLUTION OF THE TWO FIXED CENTER PROBLEM

The Hamiltonian for the two fixed center problem, obtained in the last section is

$$H = \frac{1}{2c^2}\left\{\frac{q_1^2-1}{q_1^2-q_2^2}\,p_1^2 + \frac{(1-q_2^2)}{q_1^2-q_2^2}\,p_2^2 + \frac{p_\varphi^2}{(q_1^2-1)(1-q_2^2)}\right\} - \frac{\mu}{c\,(q_1-q_2)} - \frac{\mu'}{c\,(q_1+q_2)} \tag{35}$$

The generating function $W(q_1, q_2, \varphi, P_1, P_2, P_3)$, which will ultimately be used to obtain the w_i and P_i for the perturbation equations is also a very convenient device for obtaining a direct solution to the two fixed center problem. Recalling that for the transformation to be canonical, one must have

$$\begin{aligned} p_1 &= \frac{\partial W}{\partial q_1} \\ p_2 &= \frac{\partial W}{\partial q_2} \\ p_\varphi &= \frac{\partial W}{\partial \varphi} \end{aligned} \tag{36}$$

and

$$Q_i = \frac{\partial W}{\partial P_i} \tag{37}$$

Replacement of p_1, p_2 and p_φ by the partials of W with respect to q_1, q_2, and φ, respectively, in Eq. (35) gives a partial differential equation for W which is separable. That is, a solution of the form

$$W = W_1(q_1, P_i) + W_2(q_2, P_i) + W_3(\varphi, P_i) \tag{38}$$

exists. It is a fairly simple matter to verify that

$$\left(\frac{dW_1}{dq_1}\right)^2 = \left(\frac{\partial W}{\partial q_1}\right)^2 = p_1^{\,2} = \frac{2c^2}{(q_1^{\,2}-1)^2} R^2(q_1)$$

$$\left(\frac{dW_2}{dq_2}\right)^2 = \left(\frac{\partial W}{\partial q_2}\right)^2 = p_2^{\,2} = \frac{2c^2}{(1-q_2^{\,2})^2} S^2(q_2) \tag{39}$$

$$\left(\frac{dW_3}{d\varphi}\right)^2 = \left(\frac{\partial W}{\partial \varphi}\right)^2 = p_\varphi^{\,2} = \alpha^2$$

where

$$R^2(q_1) = (q_1^{\,2}-1)\left(hq_1^{\,2} + \frac{\mu+\mu'}{c} q_1 - \beta\right) - \frac{\alpha^2}{2c^2} \tag{40}$$

$$S^2(q_2) = (1-q_2^{\,2})\left(-hq_2^{\,2} + \frac{\mu-\mu'}{c} q_2 + \beta\right) - \frac{\alpha^2}{2c^2} \tag{41}$$

In Equations (40) and (41), h is the constant energy of the two fixed center problem and is to be identified with the constant h of Equation (15) in the previous section. The separation constants are α and β. It is easily shown that α is the x-component of angular momentum about the line of centers. The constant β has no such simple interpretation.

At this stage everything necessary for the solution of the two fixed center problem is available; further discussion of the generating function will be deferred to the next section.

The Hamilton equations for the two fixed center problem give the time derivatives of q_1, q_2 and φ as

$$\dot{q}_1 = \frac{\partial H_1}{\partial p_1} = \frac{p_1}{c^2} \frac{q_1^{\,2}-1}{q_1^{\,2}-q_2^{\,2}}$$

$$\dot{q}_2 = \frac{\partial H_1}{\partial p_2} = \frac{p_2}{c^2} \frac{1-q_2^{\,2}}{q_1^{\,2}-q_2^{\,2}} \tag{42}$$

$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{c^2 (q_1^2 - 1)(1 - q_2^2)} \tag{42}$$

Combination of these equations with Equation (39) yields

$$\dot{q}_1 = \frac{\sqrt{2}}{c} \frac{R(q_1)}{q_1^2 - q_2^2}$$

$$\dot{q}_2 = \frac{\sqrt{2}}{c} \frac{S(q_2)}{q_1^2 - q_2^2} \tag{43}$$

$$\dot{\varphi} = \frac{\alpha}{c^2 (q_1^2 - 1)(1 - q_2^2)}$$

A preferable form for these equations is the following in which a parameter u is introduced which completes the separation of the variables:

$$\frac{dq_1}{R} = \frac{dq_2}{S} = du \tag{44}$$

$$dt = \frac{c}{\sqrt{2}} (q_1^2 - q_2^2)\, du \tag{45}$$

$$d\varphi = \frac{\alpha}{c\sqrt{2}} \left[\frac{1}{q_1^2 - 1} + \frac{1}{1 - q_2^2} \right] du \tag{46}$$

From Equation (44), which leads to elliptic integrals of the first kind, q_1 and q_2 turn out to be expressible as elliptic functions of u. Using these expressions for q_1 and q_2 in Equations (45) and (46), it is then possible to obtain t and φ as functions of u. The integration of Equations (45) and (46) involves elliptic integrals of the second and third kinds.

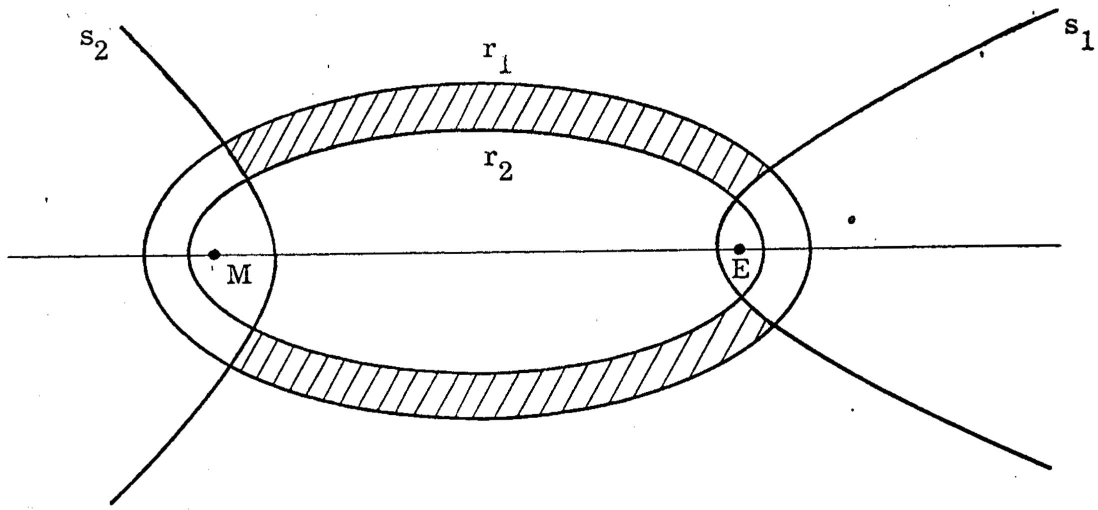
The form of solution of Equation (44) depends on the nature of the roots of the quartic expressions R^2 and S^2. These roots are uniquely determined by the three dynamical constants h, α and β. It is shown in Reference 6 that if $h < 0$, R^2 must have four real roots, two of which exceed unity and the other two lie in the interval (± 1). Further, R^2 is positive between the largest roots and also between the smallest. Since, however, q_1 must exceed unity, it follows that q_1 is constrained between the largest roots. Thus, if the roots of R^2 in order of decreasing magnitude are denoted by r_1, r_2, r_3, r_4 it may be said that

$$-1 < r_4 < r_3 < 1 < r_2 < q_1 < r_1 \tag{47}$$

This conclusion may be stated a little differently: the bounds r_1 and r_2 on q_1 represent two ellipsoids (the larger corresponding to r_1) which bound the region in space in which the vehicle may move.

The corresponding results for the quartic S^2 are more complicated: none of the roots exceed unity and at least two lie in the interval (± 1). The other two may also lie in this interval, may be real and both less than -1, or may be complex. The quartic is positive between the two largest roots and between the two smallest, if they are real. Since q_2 must lie in the interval (± 1) it follows that the orbit is constrained between the two largest roots or between the two smallest if they also lie in the (± 1) interval. If all four roots of S^2 are in (± 1), knowledge of the position of one point of the orbit specifies whether q_2 is constrained between the largest or the smallest roots; transitions from one band to the other cannot occur, since if S^2 becomes negative, $\dot{q}_2$ becomes imaginary. The roots of S^2 in the interval (± 1) correspond to hyperboloids bounding the motion in space.

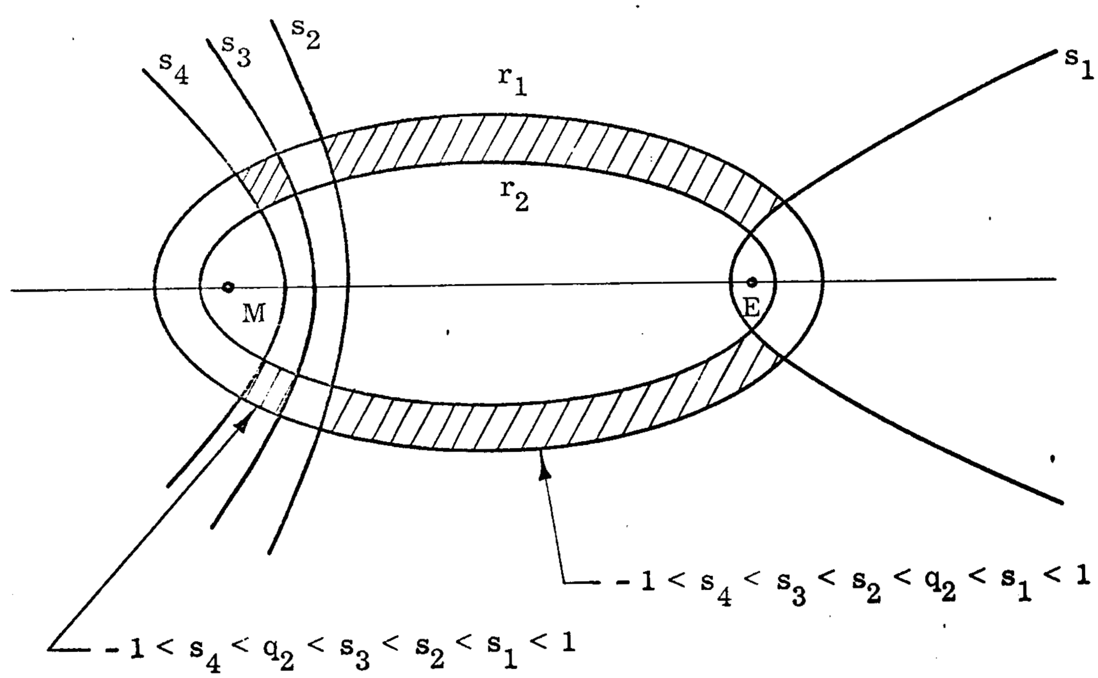
Summarizing the above results for negative energy, two possibilities for bounds on the orbit occur. These are shown in Figures II and III where the shaded areas are regions in which motion may occur.



$-1 < r_4 < r_3 < 1 < r_2 < q_1 < r_1$

$-1 < s_2 < q_2 < s_1 < 1 \begin{cases} \text{either} & s_3,\ s_4 < -1 \\ \text{or} & s_3,\ s_4 \text{ complex} \end{cases}$

Figure II



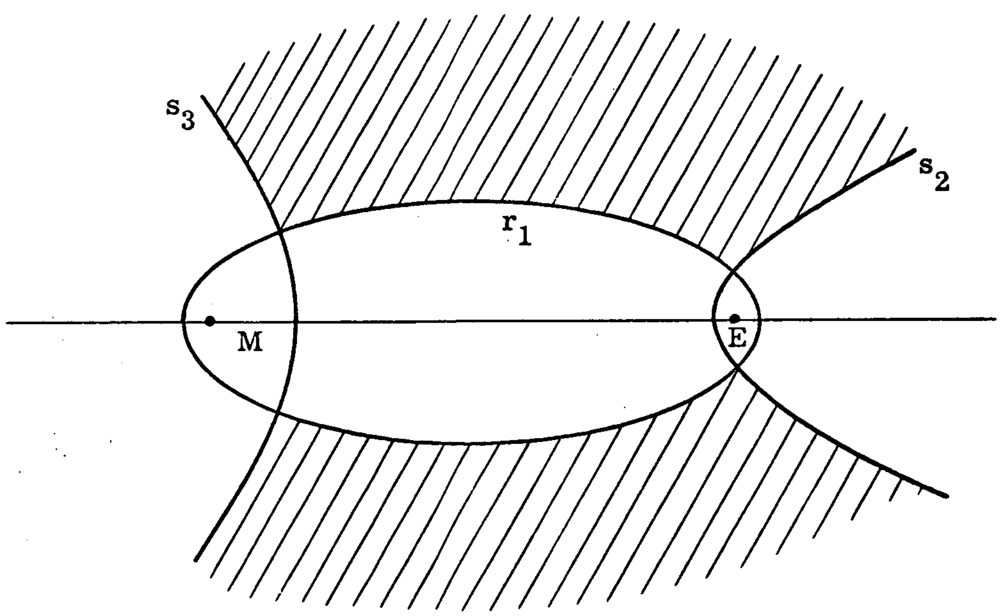
$-1 < r_4 < r_3 < 1 < r_2 < q_1 < r_1$

Figure III

If one thinks of h, α and β, which determine all the roots of R^2 and S^2 as being three dynamical specifications of a two fixed center orbit, it is clear that any remaining specifications must not violate the bounds on the region in which the motion can occur. That is, these bounds impose constraints on any further specifications. Actually, not even h, α and β can be arbitrarily selected: they must lead to roots of R^2 satisfying Equation (47) and roots of S^2 satisfying one or the other of the following:

$$
\begin{aligned}
&\text{(a)} \quad -1 \le s_2 \le s_1 \le 1 \text{ and either } s_3,\ s_4 < 1 \text{ or } s_3,\ s_4 \text{ complex} \\
&\text{(b)} \quad -1 \le s_4 \le s_3 \le s_2 \le s_1 \le 1
\end{aligned}
\qquad (48)
$$

If the energy is positive, it may be shown that R^2 has one root, say r_1 exceeding unity and is positive for q_1 exceeding r_1. The other roots are all less than 1. The quartic S^2 has two roots $s_3 < s_2$ in the interval ($\pm$1), and one on each side of this interval. It is positive for $s_3 < q_2 < s_2$. Thus in this case the motion must take place in the unbounded region shown in Figure IV.



$q > r_1 > 1$

$-1 < s_3 < q_2 < s_2 < 1$

Figure IV

As noted above, q_1 and q_2 are expressible in terms of elliptic functions of u. The particular elliptic function occurring depends on the nature of the roots. In all cases, see Reference 6,

$$q_i = \frac{A_i f(\alpha_i (u + \beta_i)) + B_i}{C_i f(\alpha_i (u + \beta_i)) - 1} \tag{49}$$

The A_i, B_i, α_i are constants depending only on the roots and hence on h, α and β. The constants β_i depend on h, α and β as well as whatever additional specifications are made to select a particular orbit. For q_1, the function f is an sn or dn function according as h is negative or positive. For q_2, $h < o$, f is an sn or cn function according as all four or only two of the roots are real and if $h > o$, f is a dn function. It is evident, of course, that q_1 and q_2 are individually periodic in the variable u. The periods of q_1 and q_2 are, however, in general non-commensurable, so that the motion in space of the vehicle will, in general, be nonperiodic. The quarter periods of q_1 and q_2 are usually denoted by K_1 and K_2, respectively, and it may be shown that these quarter periods depend only on the roots of R^2 and S^2, respectively, and hence only on h, α and β. From the way in which the β_i occur in Equation (49), it is evident that they represent a phase. In fact, it is assumed in Equation (49) that $u = o$ corresponds to some point on the orbit, say the initial point, and the β_i represent the variation in u required to get from this point to one of the extreme values of q_i - that is, to a point of tangency with one of the bounding ellipsoids for q_1, and with one of the bounding hyperboloids for q_2.

The integration of the equations for time and φ leads in all cases to the following forms (consult Reference 6)

$$t = (n_1 - n_2) u + F_1(u) + F_2(u) \tag{50}$$

$$\varphi = (m_1 + m_2) u + G_1(u) + G_2(u) \tag{51}$$

where n_1 and m_1 are constants depending on the roots of R^2, and n_2 and m_2 depend on the roots of S^2. For negative h, the functions $F_1(u)$ and $G_1(u)$ are periodic functions of u with period $2K_1$, while $F_2(u)$ and $G_2(u)$ are periodic with period $2K_2$. For positive h, the functions F_i and G_i become logarithmic.

SECTION IV - DETERMINATION OF THE GENERATING FUNCTION

The differential equations for the generating function, Eqs. (39), may be written

$$\frac{dW_1}{dq_1} = \frac{\partial W}{\partial q_1} = \frac{\sqrt{2}\,c}{q_1^{\,2}-1} \quad R$$

$$\frac{dW_2}{dq_2} = \frac{\partial W}{\partial q_2} = \frac{\sqrt{2}\,c}{1-q_2^{\,2}} \quad S \tag{52}$$

$$\frac{dW_3}{d\varphi} = \frac{\partial W}{\partial \varphi} = \alpha$$

These are ordinary differential equations, and integration again leads to elliptic integrals. Before carrying out the integration, however, some discussion of the limits on the integrals is necessary. It will be recalled that the generating function was to be a function of six variables.

$$W = W\,(q_1,\ q_2,\ \varphi,\ P_1,\ P_2,\ P_3) \tag{53}$$

and the differential equations (52) give only three of the six partial derivatives of W. Now the dependence of W on q_1, q_2 and φ can be carried by the upper limits of the integrals resulting from Eqs. (52). These upper limits should be simply q_1, q_2, and φ, respectively. Recalling further that the momenta P_i are supposed to be constants, and noting that three independent constants h, α and β already are explicitly in Eq. (52), it is evident that these three constants or some three independent functions of them must be identified with P_i. It is convenient at present to identify h, α and β themselves with P_i and defer to a later stage in the development any more complicated identification. If this is done, it now becomes obvious that the lower limits on the integrals must be either functions of h, α and β or absolute constants. This is so first because W is a function only of q_1, q_2, φ

and the P_i, and, since the integrals will be functions of their limits, only these quantities and absolute constants may be included in the limits. Secondly, the upper limits have already been taken as q_1, q_2, and φ, and recalling that the partials of W with respect to q_1, q_2 and φ must be p_1, p_2 and p_φ, no further dependence of W on q_1, q_2 and φ can be allowed without modifying the p's from which the equations (52) for W were obtained in the first place. The only remaining problem, then, is to select lower limits which depend only on h, α and β. For the integral for W_1, the variable is q_1 which has bounds on its variation. The bounds depend on h, α and β, and since r_1 is a bound whether the energy is positive or negative, it is a satisfactory lower limit. For W_2 the bounds vary with the particular conditions of the problem. However, for orbits approaching both Earth and Moon, the bound s_2 always occurs, and will be selected as the lower limit. For W_3, the situation is a little different. The variable is φ, and reference to Eq. (43) shows that φ has the sign of α and is thus monotone. Hence any absolute constant is acceptable as a lower limit and 0 will be selected. The generating function may now be written:

$$W(q_1, q_2, \varphi, h, \alpha, \beta) = W_1(q_1, h, \alpha, \beta) + W_2(q_2, h, \alpha, \beta) + W_3(\varphi, h, \alpha, \beta)$$

$$= \sqrt{2}\, c \int_{r_1}^{q_1} \frac{R}{q_1^2 - 1}\, d q_1 + \sqrt{2}\, c \int_{s_2}^{q_2} \frac{S}{1 - q_2^2}\, d q_2 + \alpha \varphi \qquad (54)$$

where W_3 is integrable directly. It might be remarked at this stage that there is an essential difference between this generating function and the corresponding function for the Kepler problem. The upper limits in the integral occurring in both generating functions may be regarded as the coordinates of a point on the orbit. In the Kepler problem, the lower limits correspond to the perigee distance for the radial integral and to zero for the two angle integrals. This may be regarded as a point on any orbit, since the angles may just be measured from the perigee point. In the two fixed center problem however, the lower limits r_1, s_2 and 0 may be regarded as a point only on a very special orbit -- namely, one which is simultaneously tangent to the ellipsoid r_1 and the hyperboloid s_2, and this tangency must occur in the x-y plane.

To complete the canonical transformation generated by W, the P_i will be identified with h, α and β as follows:

$$
\begin{aligned}
P_1 &= P_h = h \\
P_2 &= P_\beta = \beta \\
P_3 &= P_\alpha = \alpha
\end{aligned}
\tag{55}
$$

The conjugate coordinates Q_i then become

$$
\begin{aligned}
Q_1 &= Q_h = \frac{\partial W}{\partial h} = \frac{c}{\sqrt{2}} \int_{r_1}^{q_1} \frac{q_1^2\, dq_1}{R} - \frac{c}{\sqrt{2}} \int_{s_2}^{q_2} \frac{q_2^2\, dq_2}{S} \\
Q_2 &= Q_\beta = \frac{\partial W}{\partial \beta} = -\frac{c}{\sqrt{2}} \int_{r_1}^{q_1} \frac{dq_1}{R} + \frac{c}{\sqrt{2}} \int_{s_2}^{q_2} \frac{dq_2}{S} \\
Q_3 &= Q_\alpha = \frac{\partial W}{\partial \alpha} = -\frac{\sqrt{2}\,\alpha}{c} \int_{r_1}^{q_1} \frac{dq_1}{(q_1^2 - 1) R} - \frac{\sqrt{2}\,\alpha}{c} \int_{s_2}^{q_2} \frac{dq_2}{(1 - q_2^2) S} + \varphi
\end{aligned}
\tag{56}
$$

In differentiating the integrals in W there are really three terms for each integral: one is the integral of the derivative of the integrand and the other two are obtained by evaluating the integrand at the limits and multiplying by the derivatives of the limits. The terms corresponding to the limits vanish, because the upper limits are not functions of h, α and β, the integrands for the q_1 and q_2 integrals vanish at the lower limits, and the lower limit of the φ integral is an absolute constant.

It will be noted that all the integrals occurring in Eq. (56) have forms identical with one or another of those occurring in Eqs. (44), (45) and (46) for the determination of q_1, q_2, t and φ as functions of u. The only difference is that in reference 6, where the integration of Eqs. (44), (45) and (46) is carried out in all detail, the lower limit on u was taken as zero. Here the lower limits are roots of R^2 and S^2.

Of the three Q_i, Q_β has a relatively simple interpretation if one replaces dq_1 and dq_2 by du in accordance with Eq. (44). Then Q_β becomes

$$Q_\beta = -\left[\frac{c}{\sqrt{2}} \int_{u(r_1)}^{u(q_1)} du - \int_{u(s_2)}^{u(q_2)} du\right] \tag{57}$$

$$= \frac{c}{\sqrt{2}} \left(u\,(r_1) - u\,(s_2)\right)$$

since the upper limits correspond to a point on the orbit and therefore represent the same value of u. Thus Q_β appears proportional to the variation in u associated with a transit from tangency with a hyperboloid to tangency with an ellipsoid. Since the orbit is not, in general, periodic this statement does not yet uniquely define Q_β. To arrive at such a definition, it may be noted that in terms of the canonical variables P_i and Q_i the Hamiltonian becomes

$$H = h = P_1 \tag{58}$$

so that the Hamilton equations in these variables are:

$$\dot{P}_1 = \dot{P}_2 = \dot{P}_3 = \dot{h} = \dot{\alpha} = \dot{\beta} = 0 \tag{59}$$

and

$$\dot{Q}_\alpha = \dot{Q}_\beta = 0, \quad \dot{Q}_h = 1 \tag{60}$$

therefore

Q_α and Q_β are constants and

$$Q_h = t + \text{const} = t + C \tag{61}$$

The values of h, α and β may be obtained from a set of initial conditions. The values of Q_α, Q_β and C may be obtained from the initial conditions also, provided it is agreed that the $q_1 = r_1$ and $q_2 = s_2$ are to be associated, say, with the tangencies to the ellipsoid r_1 and the hyperboloid s_2 closest to the initial point. Other identifications of $q_1 = r_1$ and $q_2 = s_2$ will lead to Q's differing from those just defined by multiples of the periods K_1 and K_2.

If one applies the same analysis to Q_h and Q_α as used for Q_β (replacing dq_1 and dq_2 by u), the following expressions are obtained:

$$Q_h = t + \frac{c}{\sqrt{2}} \left[\int_{u(r_1)}^{0} q_1^2 \, du - \int_{u(s_2)}^{0} q_2^2 \, du \right] \tag{62}$$

or

$$C = \frac{c}{\sqrt{2}} \left[\int_{u(r_1)}^{0} q_1^2 \, du - \int_{u(s_2)}^{0} q_2^2 \, du \right] \tag{63}$$

and

$$Q_\alpha = - \frac{\sqrt{2}\,\alpha}{c} \left[\int_{u(r_1)}^{0} \frac{du}{q_1^2 - 1} - \int_{u(s_2)}^{0} \frac{du}{1 - q_2^2} \right] \tag{64}$$

SECTION V - ACTION AND ANGLE VARIABLES

The action and angle variables are conventionally defined only for conditionally periodic systems, which means that for the two fixed center problem the development can be made only for $h < 0$. The action variables are defined in terms of the generating function W, as follows:

$$J_1 = \oint \frac{\partial W}{\partial q_1} \, d\,q_1 = \sqrt{2}\, c \oint \frac{R \, d\,q_1}{q_1^2 - 1}$$

$$J_2 = \oint \frac{\partial W}{\partial q_2} \, d\,q_2 = \sqrt{2}\, c \oint \frac{S \, d\,q_2}{1 - q_2^2} \tag{65}$$

$$J_3 = \int_0^{2\pi} \frac{\partial W}{\partial q} \, d\varphi = 2\,\pi\,\alpha$$

where the integral for J_1 is taken over a complete cycle of variation of q_1 - i.e. from r_1 to r_2 and back to r_1, while that for J_2 is over a complete cycle of J_2 from s_1 to s_2 and back to s_1. These integrals can, for the most part, be reduced to the forms already encountered as follows:

$$J_1 = \sqrt{2}\, c \oint \frac{R \, d\,q_1}{q_1^2 - 1} = \sqrt{2}\, c \oint \frac{R^2}{q_1^2 - 1} \, \frac{d\,q_1}{R}$$

$$= \sqrt{2}\, c \int_0^{4K_1} \left[h\,q^2 + \frac{\mu + \mu'}{c}\, q_1 - \beta - \frac{\alpha^2}{2c^2\,(q_1^2 - 1)} \right] du \tag{66}$$

A complete cycle of variation q_1 corresponds to a variation in u of $4 K_1$. Now the first term in this integral has the form of the dependence of the time on q_1, and, referring to Eq. (50) it is seen that the periodic part F_1 will vanish and hence the contribution of the first term to the integral is $8 h n_1 K_1$. Similarly the last term has the form of the q_1 part of the φ integral, Eq. (51), and will contribute $- \alpha \cdot 4m_1 K_1$. The β term contributes just $-\sqrt{2}\, c\, \beta \cdot 4 K_1$. The only new integral to evaluate is

$$\int_0^{4K_1} q_1 \, du \tag{67}$$

This integral, too, turns out to be expressible as a linear term in u plus a periodic one, so that for the limits given, it contributes a term $\sqrt{2}\,(\mu + \mu')\, \ell_1 \cdot 4 K_1$ where ℓ_1 is the coefficient of the linear term. Thus, finally,

$$J_1 = 8\, h\, n_1 K_1 + 4\sqrt{2}\,(\mu + \mu')\, \ell_1 K_1 - 4\sqrt{2}\, c\, \beta K_1 - 4\, \alpha\, m_1 K_1 \tag{68}$$

In an exactly similar fashion

$$J_2 = -8\, h\, n_2 K_2 + 4\sqrt{2}\,(\mu - \mu')\, \ell_2 K_2 + 4\sqrt{2}\, c\, \beta K_2 - 4\, \alpha\, m_2 K_2 \tag{69}$$

To obtain the angle variables conjugate to the action variables, it is necessary to recall that the original condition imposed on the P_i was only that they be constants. Identification of the P_i with h, α, and β is only one possibility; any three independent functions of h, α, and β would serve as well and, in particular, it is now desirable to identify P_i with J_i. Now the generating function W is given in Eq. (54) in terms of q_1, q_2, φ, h, α, β, and r_1 and s_2. The roots r_1 and s_2 are, however, functions of h, α and β. Now if Eqs. (68) and (69) together with the third of Eqs. (65) be inverted to express h, α, and β in terms of J_1, J_2, and J_3, it will be possible to substitute for h, α, and β in W to obtain W as a function of q_1, q_2, φ, J_1, J_2, and J_3. It should be remarked that the inversion to obtain h, α, and β

in terms of J_1, J_2 and J_3 is not an easy task since the coefficients n_1, n_2, ℓ_1, ℓ_2, m_1, m_2 are very complicated functions of h, α and β. Nevertheless the procedure is possible in principle and the angle variables w_i conjugate to the J's are given by the partial derivatives of the generating function W with respect to the J's:

$$w_i = \frac{\partial W}{\partial J_i} \tag{70}$$

One may obtain expressions for the w_i without actually performing the inversion, by writing the derivatives of W with respect to J_i in terms of its derivatives with respect to h, α, and β:

$$\begin{aligned} w_i &= \frac{\partial W}{\partial J_i} = \frac{\partial W}{\partial h}\frac{\partial h}{\partial J_i} + \frac{\partial W}{\partial \alpha}\frac{\partial \alpha}{\partial J_i} + \frac{\partial W}{\partial \beta}\frac{\partial \beta}{\partial J_i} \\ &= Q_h \frac{\partial h}{\partial J_i} + Q_\alpha \frac{\partial \alpha}{\partial J_i} + Q_\beta \frac{\partial \beta}{\partial J_i} \end{aligned} \tag{71}$$

from Eqs. (56) defining the variables conjugate to h, α and β. Or, recalling Eq. (62) for Q_h,

$$w_i = (t + C)\frac{\partial h}{\partial J_i} + Q_\beta \frac{\partial \beta}{\partial J_i} + Q_\alpha \frac{\partial \alpha}{\partial J_i} \tag{72}$$

where C, Q_α and Q_β are constants.

The derivatives of h, α and β may be expressed in terms of the n's, m's, ℓ's and K's occurring in Eqs. (68) and (69) by first obtaining the partials of the J's with respect to h, α and β from Eqs. (65), and then inverting their Jacobian matrix. The results of this calculation for the Jacobian are

$$J\begin{pmatrix} J_1 J_2 J_3 \\ h\ \beta\ \alpha \end{pmatrix} = \begin{bmatrix} 4 n_1 K_1 & -2\sqrt{2}\, c K_1 & -4 m_1 K_1 \\ -4 n_2 K_2 & 2\sqrt{2}\, c K_2 & -4 m_2 K_2 \\ 0 & 0 & 2\pi \end{bmatrix} \tag{73}$$

and its inverse is

$$J\begin{pmatrix} h & \beta & \alpha \\ J_1 & J_2 & J_3 \end{pmatrix} = \begin{bmatrix} \dfrac{1}{4K_1(n_1 - n_2)} & \dfrac{1}{4K_2(n_1 - n_2)} & \dfrac{m_1 + m_2}{2\pi\,(n_1 - n_2)} \\[2ex] \dfrac{\sqrt{2}\, n_2}{4cK_1(n_1 - n_2)} & \dfrac{\sqrt{2}\, n_1}{4cK_2(n_1 - n_2)} & \dfrac{\sqrt{2}\,(n_1 m_2 - m_1 n_2)}{2\pi\, c(n_1 - n_2)} \\[2ex] 0 & 0 & \dfrac{1}{2\pi} \end{bmatrix} \tag{74}$$

so that, finally

$$\begin{aligned} w_1 &= \frac{t + C}{4K_1\,(n_1 - n_2)} + \frac{\sqrt{2}\; n_2\, Q_\beta}{4cK_1\,(n_1 - n_2)} \\ w_2 &= \frac{t + C}{4K_2\,(n_1 - n_2)} + \frac{\sqrt{2}\; n_1\, Q_\beta}{4cK_2\,(n_1 - n_2)} \\ w_3 &= \frac{(t + C)\,(m_1 + m_2)}{2\pi\,(n_1 - n_2)} + \frac{\sqrt{2}\; Q_\beta\,(n_1 m_2 + m_1 n_2)}{2\pi\, c\,(n_1 - n_2)} + \frac{Q_\alpha}{2\pi} \end{aligned} \tag{75}$$

are the angle variables.

SECTION VI - CONCLUSION

To complete the solution of the restricted problem, it is now necessary to express the disturbing function H_2 in terms of the action and angle variables. This is a formidable problem. The disturbing function is given in terms of q_1, q_2, φ and their conjugate momenta in Eq. (34). The momenta are given in terms of q_1, q_2, φ, h, α and β by Eqs. (39) so that H_2 may readily be written in terms of these variables. Starting from the other end, the action variables J_1 and J_2 are given in terms of complicated functions of h, α, and β [Eqs. (68) and (69)] while J_3 is just $2\pi\alpha$ [Eqs. (65)]. The angle variables w_i are given by Eq. (75) as linear functions of Q_h, Q_α, and Q_β with coefficients which are functions of h, α, and β similar to those occurring for J_i. And Q_h, Q_α, and Q_β are related to q_1, q_2, φ, and h, α, and β by Eqs. (56). Thus, the following procedure would yield the information necessary to write H_2 (w_i, J_i):

1. Express K_1, K_2, ℓ_1, ℓ_2, n_1, n_2, m_1, m_2 as functions of h, α, and β.

2. $$\alpha = \frac{J_3}{2\pi}$$

   Invert Eqs. (68) and (69) using the results of step 1 to obtain $h(J_i)$ and $\beta(J_i)$.

3. Express K_1, K_2, ℓ_1, ℓ_2, n_1, n_2, m_1, m_2 which are functions of h, α and β in terms of J_i.

4. Invert Eqs. (75) to obtain $Q_h = t + c$, Q_α and Q_β as functions of the angle variables w_i and K_1, K_2, ℓ_1, ℓ_2, n_1, n_2, m_1, and m_2.

5. Use step 1 to obtain Q_h, Q_α, and Q_β as functions of w_i and J_i.

6. Invert Eqs. (56) to obtain q_1, q_2, and φ as functions of Q_h, Q_α, Q_β, h, α, and β.

7. In the expressions for q_1, q_2, and φ obtained in step 6 replace Q_h, Q_α, and Q_β using step 5 and h, α, and β using step 2 to obtain q_1, q_2, and φ in terms of w_i and J_i.

8. In the disturbing function H_2 (q_1, q_2, φ, h, α, β), replace q_1, q_2, and φ from step 7 and h, α, and β from step 2 to obtain, finally, H_2 (w_i, J_i).

Steps 1, 2, and 6 are the difficult ones in this procedure. It is relatively easy to write K_1, K_2, ℓ_1, ℓ_2, n_1, n_2, m_1, and m_2 as functions of the roots of the quartics and two intermediate parameters which are related to the roots of the quartics by transcendental equations. The roots of the quartics are, of course, functions of h, α, and β, but it is not easy to write out these functions explicitly. Thus, even step 1 is quite difficult, and to perform the inversion required in step 2 in closed form appears nearly impossible.

It should be remarked, however, that, at least for certain types of orbits, it should be possible to get fairly good approximations of these steps. For a lunar orbit which starts from the earth, closely circles the moon and returns to the earth, it may be shown that $\alpha^2/2c^2$ is very small. This is so because such an orbit has very close approaches to the line of centers, and recalling that α is the angular momentum about the line of centers, it follows that α must be small. If α were zero, two of the roots of the quartics would be ± 1 and the other two are obtained in terms of h and β by solving quadratics [see Eqs. (40) and (41)]. Now it is possible to obtain the roots of the quartics for small α in terms of those for zero α in a series of powers of α. Thus for small α, it is easy to obtain fairly simple approximate expressions for the roots in terms of h, α, and β. Further, it turns out that the transcendental equations to be inverted for the intermediate parameters are very well approximated by just two terms of an expansion. Thus, it is feasible, for lunar orbits, to obtain a good approximation to steps 1 and 2.

The complete elliptic integrals

$$\oint q_1^{\,2}\, du, \qquad \oint \frac{d\,q_1}{q_1^{\,2} - 1}, \qquad \oint dq_1$$

and similar ones for q_2, have forms very similar to those obtained by Vinti[8] in his model for the oblate earth. Vinti used oblate spheroidal coordinates for his model and the close connection between his development and that given in this report for the two fixed center problem was first pointed out by Pines[9]. The Vinti integrals have recently been evaluated approximately by Izsak[10] using a technique developed by Sommerfeld[11, 12] for evaluating certain contour integrals of functions with branch points. The method is to expand the integrals in terms of a quadratic function and evaluate the series of resulting integrals about contours enclosing the roots of the quadratic. The values of the integrals so obtained are explicitly in terms of the coefficients of the quartics. For the method to be valid, the expansion must converge over both the original and the final contours. This condition is satisfied for Izsak's expansion of the Vinti integrals. However, none of the obvious expansions for the two fixed center integrals converge over the final contour.

The greatest difficulty in following the procedure for obtaining H_2 is in step 6. Eqs. (56) relating Q_h, Q_α, and Q_β with q_1, q_2, and φ are transcendental equations and it is hard to say how well their inversion could be approximated by some approximation procedure, such as the Lagrange inversion theorem.

It should be remarked that it would be possible to write H_2 in terms of Q_h, Q_α, Q_β, h, α, and β rather than in terms of w_1 and J_1. This is not done in the Kepler problem because the relation between the original coordinates and time is best achieved by a Fourier expansion in the mean anomaly rather than in time. An expansion in time would involve far more complicated coefficients. Which set of variables will turn out to be better for the two fixed center problem is hard to predict at this stage.

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